Multiparametric generalized algebras and applications

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Summary

These multiparametric algebras extend the structure of some types of generalized functions developed over the past 20 years ago for many authors.
They are constructed by means of some independant parameters, a sheaf of topological algebras and a factor ring linked to the asymptotic structure adjustable to each problem to solve. We give some examples of studying and solving some differential problems with several independent singularities as non Lipschitzian nonlinearity, characteristic cases, irregular data.
The *singular support* of the solution localizes its singularities. The *singular spectrum* allows a spectral analysis of these songularities and its linear, differential and nonlinear properties are studied with examples in some cases.

First Part: Algebraic structure and toolbox designed to sove very irregular differential problem

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- A and I are both solid, i.e., equal to their solid hull,

$$\mathrm{sh}(A) = \left\{ x \in \mathbb{K}^{\Lambda} \mid \exists a \in A,\ \forall \lambda \in \Lambda : |x_\lambda| \leq |a_\lambda| \right\} .$$

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- $(\mathcal{E}, \mathcal{P})$ is a sheaf of topological $\mathbb{K}$–algebras on a topological space X, the topology on $\mathcal{E}(\Omega)$ being given, for any open set Ω in X, by a family $\mathcal{P}(\Omega)$ of semi-norms.

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- Then we set

$$\mathcal{H}_{(A,\mathcal{E},\mathcal{P})}(\Omega) = \left\{ (u_\lambda)_\lambda \in [\mathcal{E}(\Omega)]^\Lambda \mid \forall p \in \mathcal{P}(\Omega), (p(u_\lambda))_\lambda \in A \right\} ,$$

$$\mathcal{J}_{(I,\mathcal{E},\mathcal{P})}(\Omega) = \left\{ (u_\lambda)_\lambda \in [\mathcal{E}(\Omega)]^\Lambda \mid \forall p \in \mathcal{P}(\Omega), (p(u_\lambda))_\lambda \in I \right\} .$$

Under some more technical conditions detailed in some papers cited in the bibliography we have the theorem

- **Theorem 1** *The factor space* $\mathcal{A} = \mathcal{H}_{(A,\mathcal{E},\mathcal{P})} / \mathcal{J}_{(I,\mathcal{E},\mathcal{P})}$ *is a presheaf with localisation principle. Moreover if* $\mathcal{E}$ *is a fine sheaf,* $\mathcal{A}$ *is also a fine sheaf.* $\mathcal{A}$ *is said a presheaf of* $(\mathcal{C},\mathcal{E},\mathcal{P})$*-algebras. The equivalence class in* $A(\Omega)$ *of* $(u_\lambda)_{\lambda\in\Lambda} \in \mathcal{H}_{(A,\mathcal{E},\mathcal{P})}(\Omega)$ *is denoted by* $[u_\lambda]$.

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- **Definition 1** *Consider a family* B *of nets in* $(\mathbb{R}_+^*)^\Lambda$*. Let* $\langle B\rangle$ *be the closure of* B *in* $(R_+^*)^\Lambda$ *with respect to addition and division;* $\langle B\rangle$ *is given as the subset of elements in* $(\mathbb{R}_+^*)^\Lambda$ *obtained as rational fractions of elements in* B *with coefficients in* N^**, Now let*
$$A = \left\{ (a_\lambda)_\lambda \in \mathbb{K}^\Lambda \mid \exists\, (b_\lambda)_\lambda \in \langle B\rangle, \exists \lambda_0 \in \Lambda, \forall \lambda \prec \lambda_0 : |a_\lambda| \leq b_\lambda \right\}.$$
Then we say that A **is overgenerated by** B. *An ideal of* A *is given by*
$$I_A = \left\{ (a_\lambda)_\lambda \in \mathbb{K}^\Lambda \mid \forall\, (b_\lambda)_\lambda \in B, \exists \lambda_0 \in \Lambda, \forall \lambda \prec \lambda_0 : |a_\lambda| \leq b_\lambda \right\}.$$

Embedding and association

- If Ω is open subset of $\mathbb{R}^n$, $\mathcal{D}'(\Omega)$ can be embedded into $\mathcal{A}(\Omega)$ via the standard mollifier φ_λ when choosing A overgenerated by some B of $(\mathbb{R}_+^*)^{]0,1]}$ containing the family $(\lambda)_\lambda$.

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- **Definition 2** The association process
  Let $a : \Lambda \to K$ *with* $(a(\lambda))_\lambda = (a_\lambda)_\lambda \in A$ *and* E *a given sheaf of topological* $K-$*vector space containing* E *as a subsheaf and* $J_{(I_A,\mathcal{E},\mathcal{P})}(\Omega) \subset \left\{ (u_\lambda)_\lambda \in \mathcal{H}_{(A,\mathcal{E},\mathcal{P})}(\Omega) \lim\limits_{E(\Omega),\Lambda} u_\lambda = 0 \right\}$. *We say that* $u = [u_\lambda]$ *and* $v = [v_\lambda] \in A(\Omega)$ *are* $a - E$ *associated if* $\lim\limits_{E(\Omega),\Lambda} a_\lambda (u_\lambda - v_\lambda) = 0$ *and write* $u \underset{E(\Omega)}{\overset{a}{\sim}} v$

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- An association process can also been defined between $u = [u_\lambda]$ and $T \in E(\Omega)$: $u \sim T \Longleftrightarrow \lim\limits_{E(\Omega),\Lambda} u_\lambda = T$. Taking $E = \mathcal{D}'$, $\mathcal{E} = \mathrm{C}^\infty$, $\Lambda =]0,1]$, we recover the association process defined in the literature by Colombeau and others

Solving a singular Cauchy problem

We study the following Cauchy problem

$$(P)\begin{cases} \dfrac{\partial^2 u}{\partial x \partial y} = F(.,.,u) \\ u\mid_\gamma = \varphi \\ \dfrac{\partial u}{\partial y}\mid_\gamma = \psi \end{cases}$$

where γ is the curve of equation $y = f(x)$, φ and ψ are the Cauchy data

specified later, F may be non Lipschitz in u. Moreover other types of singularities are possible, the curve γ may be characteristic or data non regular

We replace (P) by a family $(P_{(\varepsilon,\eta,\rho)})$ of regularized problems

$$(P_{\varepsilon,\eta,\rho})\left\{\begin{array}{c} \dfrac{\partial^2 u_{\varepsilon,\eta,\rho}}{\partial x \partial y} = F_\varepsilon(.,.,u) \\ u_{\varepsilon,\eta,\rho}(x, f_\eta(x)) = \varphi_\rho(x) \\ \dfrac{\partial u_{\varepsilon,\eta,\rho}}{\partial y}(x, f_\eta(x)) = \psi_\rho(x) \end{array}\right.$$

where F_ε is Lipschitz, f_η non characteristic and φ_ρ and ψ_ρ regular.

Let $\left(H_{\varepsilon,\eta,\rho}\right)$ the assertion

$$\left(H_{\varepsilon,\eta,\rho}\right)\left\{\begin{array}{c} f_\eta \in \mathrm{C}^\infty(\mathbb{R}),\ f'_\eta > 0, f_\eta(\mathbb{R}) = \mathbb{R} \\ F_\varepsilon \in \mathrm{C}^\infty\left(\mathbb{R}^3, \mathbb{R}\right), \forall K \Subset \mathbb{R}^2, \\ \sup\limits_{(x,y)\in K, z\in\mathbb{R}} \left|\partial_z F_\varepsilon(x,y,z)\right| = m_{K,\varepsilon} <= +\infty \\ \varphi_\rho \text{ and } \psi_\rho \in \mathrm{C}^\infty(\mathbb{R}) \end{array}\right.$$

- **Theorem 2** *Under assumption* $\left(H_{\varepsilon,\eta,\rho}\right)$, *Problem* $\left(P_{\varepsilon,\eta,\rho}\right)$ *has a unique solution in* $\mathrm{C}^{\infty}\left(\mathbb{R}^{2}\right)$. For detailed proof, see the bibliography. One can prove that $\left(P_{\varepsilon,\eta,\rho}\right)$ is equivalent to the integral formulation

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$$\left(I_{\varepsilon,\eta,\rho}\right) : u_{\varepsilon,\eta,\rho}(x,y) = u_{0,\varepsilon,\eta,\rho}(x,y) - \iint_{D(x,y,f_\eta)} F_\varepsilon \xi, \zeta, u_{\varepsilon,\eta,\rho}(\xi,\zeta) d\xi d\zeta$$

where

$$D(x,y,f_\eta) = \begin{cases} \left\{(\xi,\zeta) : f_\eta^{-1}(y) \leq \xi \leq x, y \leq \zeta \leq f_\eta(\xi)\right\} & \text{if } y \leq \ f_\eta (\\ \left\{(\xi,\zeta) : x \leq \xi \leq f_\eta^{-1}(y), f_\eta(\xi) \leq \zeta \leq y\right\} & \text{if } y \geq \ f_\eta \end{cases}$$

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- The main idea consists to a Picard's procedure to define a sequence of successive approximations

$$u_{n,\varepsilon,\eta,\rho}(x,y) = u_{0,\varepsilon,\eta,\rho}(x,y) - \iint_{D(x,y,f_\eta)} F_\varepsilon \xi, \zeta, u_{n-1,\varepsilon,\eta,\rho}(\xi,\zeta) d\xi d\zeta$$

Finally the sequence $u_{n,\varepsilon,\eta,\rho}$ converge uniformly on any compact set to $u_{\varepsilon,\eta,\rho} = u_{0,\varepsilon,\eta,\rho} + \sum_{n\geq 1} v_{n,\varepsilon,\eta,\rho}$ which verifies $\left(I_{\varepsilon,\eta,\rho}\right)$

Cut off procedure for non Lipschitz function

Let $\varepsilon \in (0,1]$ and $(r_\varepsilon)_\varepsilon \in (\mathbb{R}_+^*)^{]0,1]}$ such that $\lim\limits_{\varepsilon \to 0} r_\varepsilon = +\infty$. Consider a family of smooth one-variable function $(g_\varepsilon)_\varepsilon$ such that $\sup\limits_{z \in [-r_\varepsilon, r_\varepsilon]} |g_\varepsilon(z)| = 1$ with $g_\varepsilon(z) = 0$ if $|z| \geq r_\varepsilon$ and $g_\varepsilon(z) = 1$ if $-r_\varepsilon + 1 \leq z \leq r_\varepsilon - 1$ Assume that $\partial^n g_\varepsilon / \partial z^n$ is bounded on $[-r_{\varepsilon,}, r_\varepsilon]$ and $\sup\limits_{z \in [-r_{\varepsilon,} r_\varepsilon]} |\partial^n g_\varepsilon(z) / \partial z^n| = M_n$ let $\Phi_\varepsilon(z) = z g_\varepsilon(z)$. We approximate F by $F(x, y, \Phi_\varepsilon(z)) = F_\varepsilon(x, y, z)$.

Characteristic non Lipschitz problem with irregular data

- We choose this very irregular situation to show the efficiency of our multi parametrized algebras To give a meaning to the problem we replace it with the family of problems

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(P_{\varepsilon,\eta,\rho})\begin{cases} \dfrac{\partial^2 u_{\varepsilon,\eta,\rho}}{\partial x \partial y}(x,y) = F_\varepsilon(x,y,u_{\varepsilon,\eta,\rho}(x,y) \\ u_{\varepsilon,\eta,\rho}(x,\eta x) = \varphi_\rho(x) \\ \dfrac{\partial u_{\varepsilon,\eta,\rho}}{\partial y}(x,\eta x) = \psi_\rho(x) \end{cases}
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- where $\left(\varphi_\rho\right)_\rho$ and $\left(\psi_\rho\right)_\rho$ are representatives of φ and ψ in the algebra $\mathcal{A}(\mathbb{R})$ to see below. **The parameter η permits to replace the $x-$axis, characteristic line $y=0$ with a noncharacteristic one $y=\eta x$ on which the parameter ρ makes the data regular**

Generalized solution to Characteristic non Lipschitz problem with irregular data

- To get the result, we have to verify some hypotheses detailed in some papers given in bibliography. The main hypothese is the following

$$H \begin{cases} \mathcal{C} = A/I_A \text{ is overgenerated by the following elements of } \\ \mathbb{R}_+^{(0,1]x(0,1]x(0,1]} \\ (\varepsilon)_{(\varepsilon,\eta,\rho)}, (r_\varepsilon)_{(\varepsilon,\eta,\rho)}, (\eta)_{(\varepsilon,\eta,\rho)}, (\rho)_{(\varepsilon,\eta,\rho)}, \left(e^{r_\varepsilon^p \eta^{-1}}\right)_{(\varepsilon,\eta,\rho)} \\ \mathcal{A}\left(\mathbb{R}^2\right) = \mathcal{H}\left(\mathbb{R}^2\right)/\mathcal{J}\left(\mathbb{R}^2\right) \text{ is built on } \mathcal{C} \\ (\mathcal{E},\mathcal{P}) = \left(\mathrm{C}^\infty\left(\mathbb{R}^2\right), (P_{K,l})_{K\Subset\mathbb{R}^2, l\in\mathbb{N}}\right) \end{cases}$$

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- **Theorem 3** *If* $u_{\varepsilon,\eta,\rho}$ *is the solution to problem* $(P_{\varepsilon,\eta,\rho})$ *given in* Theorem 2**,** *then, under* $H-$*hypotheses, the family* $(u_{\varepsilon,\eta,\rho})_{(\varepsilon,\eta,\rho)}$ *is a representative of a generalized function* u *which belongs to the algebra* $\mathcal{A}\left(\mathbb{R}^2\right)$

Sketch of the proof

The assumptions ensure that
$\forall K = [-a, a]\text{x}[-a, a]\,, \forall \alpha \in \mathbb{N}^2, \left(P_{K,\alpha}\left(u_{0,\varepsilon,\eta,\rho}\right)\right)_{(\varepsilon,\eta,\rho)} \in A.$
From what we can prove that $\forall K \Subset \mathbb{R}^2, \left(P_{K,0}\left(u_{\varepsilon,\eta,\rho}\right)\right)_{(\varepsilon,\eta,\rho)} \in A$ and $\left(P_{K,1}\left(u_{\varepsilon,\eta,\rho}\right)\right)_{(\varepsilon,\eta,\rho)} \in A.$ Then, by induction we can prove that

$$\forall n \in \mathbb{N}\left(P_{K,n}\left(u_{\varepsilon,\eta,\rho}\right)\right)_{(\varepsilon,\eta,\rho)} \in A$$

consequently we have $\left(u_{\varepsilon,\eta,\rho}\right)_{(\varepsilon,\eta,\rho)} \in \mathcal{H}\left(\mathbb{R}^2\right).$
For more details, see the bibliography

Second Part: Local and Microlocal Analysis

- **Definition 3** *To any presheaf* $\mathcal{F}$ *of topological* K*–vector space (as* $\mathcal{D}'$*) which contain the subsheaf* $\mathcal{E}$ (as C^{∞}), *we associate the following subsheaf of* A,

$$\mathcal{F}_{\mathcal{A}}(\Omega) = \left\{ u \in \mathcal{A}(\Omega) \mid \exists \left(u_{\lambda}\right)_{\lambda} \in u, \ \exists f \in \mathcal{F}(\Omega) \ : \ \lim_{\Lambda} {}_{\mathcal{F}(\Omega)} \, u_{\lambda} = f \right\}$$

Localisation of singularities : F-singular support

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- This leads to the open set (of points) of $\mathcal{F}$–regularity of a given section $u \in \mathcal{A}$,

$$\mathcal{O}_{\mathcal{A}}^{\mathcal{F}}(u) = \{ x \in \Omega \mid \exists V \in \mathcal{V}_x : u|_V \in \mathcal{F}_{\mathcal{A}}(V) \},$$

where $\mathcal{V}_x$ denotes the set of all the neighborhoods of x.

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where $\mathcal{V}_x$ denotes the set of all the neighborhoods of x.

- The complement of $\mathcal{O}_{\mathcal{A}}^{\mathcal{F}}(u)$ on Ω, is called the $\mathcal{F}$–singular support of $u \in \mathcal{A}(\Omega)$:

$$\mathcal{S}_{\mathcal{A}}^{\mathcal{F}}(u) = \Omega \setminus \mathcal{O}_{\mathcal{A}}^{\mathcal{F}}(u).$$

It is clear that $\mathcal{S}_{\mathcal{A}}^{\mathcal{F}}(u)$ is a closed subset contained in $\operatorname{supp}(u)$.

Microlocalisation of F-singularities: Asymptotic singular spectrum

- The idea of the $(a, \mathcal{F})$-microlocal analysis is the following.
  If $u = [u_\lambda] \in \mathcal{A}(\Omega)$ is not in $\mathcal{F}_\mathcal{A}$ in a given point $x \in \Omega$, i.e., there is no $V \in \mathcal{V}_x$ and $f \in \mathcal{F}(V)$ such that $\lim_\Lambda u_\lambda = f$ in $\mathcal{F}(V)$, there are nevertheless generalized numbers $a \in \mathcal{C} = A/I$ "small enough" such that $a\, u \in \mathcal{F}_\mathcal{A}(V)$ in some neighborhood V of x. (In particular, this is true for $a = 0$.)

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- More precisely, we can consider a map $a : \mathbb{R}_+^n \to \mathcal{C} = A/I$ with some convenient properties, such that the set of $r \in \mathbb{R}_+^n$ for which $a(r)\,u \in \mathcal{F}_\mathcal{A}$, will give more detailed information of the "roughness" of u at a given point. These considerations lead to the following definition which generalizes the corresponding one in a previous paper where a is a map from $\mathbb{R}_+$ to A_+, instead of $\mathbb{R}_+^n$ to $\mathcal{C}$ as here:

- **Definition 4** $\mathcal{F}$ is a given sheaf of topological $\mathbb{K}$–vector spaces over X containing $\mathcal{E}$ as a subsheaf, a is a map from $\mathbb{R}_+^n$ to $\mathcal{C}$ such that $a(o) = 1_{\mathcal{C}}$, and

$\forall r, s \in \mathbb{R}_+^n \setminus \{ o \} : a(r) = o(1)$ and $a(r+s) = O(a(r)\, a(s))$where
$a(r) = [a_\lambda(r)]$, with $r = (r_1, r_2, \ldots r_n) \in \mathbb{R}_+^n$, and

$$a' = O(a) \iff \exists\, C > 0,\ (a'_\lambda) \in a',\ (a_\lambda) \in a,\ \forall \lambda \in \Lambda :\ |a'_\lambda| \leq C\, |a_\lambda|\ .$$

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- Then we define, for any open subset Ω of X, $u = [u_\lambda] \in \mathcal{A}(\Omega)$ and $x \in \Omega$, **the** $(a, \mathcal{F})$**–regular fiber of** u **over** x as the set
  $N_{(a,\mathcal{F}),x}(u) =$
  $$\left\{ r \in \mathbb{R}_+^n \mid \exists V \in \mathcal{V}_x,\ \exists f \in \mathcal{F}(V) : \lim_{\Lambda} {}_{\mathcal{F}(V)} \left(a_\lambda(r)\, u_\lambda \,|_V \right) = f \right\}$$

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  X containing $\mathcal{E}$ as a subsheaf, a is a map from $\mathbb{R}_+^n$ to $\mathcal{C}$ such that $a(o) = 1_{\mathcal{C}}$, and

$\forall r, s \in \mathbb{R}_+^n \setminus \{ o \} : a(r) = o(1)$ and $a(r+s) = O(a(r)\, a(s))$where $a(r) = [a_\lambda(r)]$, with $r = (r_1, r_2, ... r_n) \in \mathbb{R}_+^n$, and

$$a' = O(a) \iff \exists\, C > 0,\ (a'_\lambda) \in a',\ (a_\lambda) \in a,\ \forall \lambda \in \Lambda : \ |a'_\lambda| \le C\, |a_\lambda| \ .$$

- Then we define, for any open subset Ω of X, $u = [u_\lambda] \in \mathcal{A}(\Omega)$ and $x \in \Omega$, **the** $(a, \mathcal{F})$**–regular fiber of** u **over** x as the set $N_{(a,\mathcal{F}),x}(u) =$
  $$\left\{ r \in \mathbb{R}_+^n \mid \exists V \in \mathcal{V}_x,\ \exists f \in \mathcal{F}(V) : \lim_{\Lambda}{}_{\mathcal{F}(V)} \left(a_\lambda(r)\, u_\lambda \left|_V \right.\right) = f \right\}$$
- $= \{ r \in \mathbb{R}_+^n \mid \exists V \in \mathcal{V}_x : a(r)\ \ u|_V \in \mathcal{F}_{\mathcal{A}}(V) \}$

- and **the** $(a, \mathcal{F})$**–singular fiber of** u **over** x **as its complement in** $\mathbb{R}_+^n$**,**

$$\Sigma_{(a,\mathcal{F}),x}(u) = \mathbb{R}_+^n \setminus N_{(a,\mathcal{F}),x}(u).$$

- When $(a,\mathcal{F})$ can be considered to be given, we shall write for short: $N_{(a,\mathcal{F}),x}(u) = N_x(u)$, $\Sigma_{(a,\mathcal{F}),x}(u) = \Sigma_x(u)$.

In other terms, for each $r \in N_x(u)$ (resp. $r \in \Sigma_x(u)$), the generalized function u is (resp. cannot be) locally $a(r)$-associated to a section of $\mathcal{F}$ above x.

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- **Theorem 4** *The $(a,\mathcal{F})$–regular fiber $N_x(u)$ is the (possibly empty) union $\bigcup_{r \in N_x(u)} \Gamma_r$ of closed cones $\Gamma_r = \prod_{i=1}^{n} [r_i, +\infty) = r + R_+^n$ .The (a,F)–singular fiber $\Sigma_x(u)$ is either R_+^n, or $\bigcap_{r \in N_x(u)} (\mathbb{R}_+^n \setminus \Gamma_r)$*

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- **Proof** We have to show that $r \in N_x(u) \;\Rightarrow\; r + s \in N_x(u)$ for all $s \in \Gamma$. Then, with any point r, $N_x(u)$ also contains the cone $r + \Gamma$, and $N_x(u)$ equals the union of these cones as claimed. Assume that $r \in N_x(u)$, *i.e.*, $a(r)\, u|_V \in \mathcal{F}_{\mathcal{A}}(V)$ for some neighborhood V of x. For $s = o$, there is nothing to show, and for $s \neq o$, we have $a(s) = o(1)$ and thus $a(r + s) = O(a(r)\, a(s)) = o(1)\, a(r)$, whence $a(r + s)\, u|_V = o(1)\, a(r)\, u|_V \in \mathcal{F}_{\mathcal{A}}(V)$.

- **Definition 5** *The* $(a, \mathcal{F})$*-singular spectrum of* $u \in A(\Omega)$ *is the set*

$$\mathcal{S}_{\mathcal{A}}^{(a,\mathcal{F})}(u) = \left\{ (x, r) \in \Omega \times \mathbb{R}_{+}^{n} \mid r \in \Sigma_{x}(u) \right\}.$$

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- **Theorem 5** *The projection of the* $(a,\mathcal{F})$*–singular spectrum of* u *on* Ω *is the* F*–singular support of* u*,*

$$\left\{ x \in \Omega \mid \exists r \in \mathbb{R}_{+}^{n} : (x,r) \in \mathcal{S}_{\mathcal{A}}^{(a,\mathcal{F})}(u) \right\} = \mathcal{S}_{\mathcal{A}}^{\mathcal{F}}(u).$$

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- **Proof** This is an immediate consequence of previous definitions. Indeed,
$\exists r : (x, r) \in \mathcal{S}_{\mathcal{A}}^{(a,\mathcal{F})}(u) \iff \Sigma_x(u) \neq \emptyset \iff o \notin N_x(u) \iff u|_V \notin \mathcal{F}_{\mathcal{A}}(V)$.

- **Theorem 6** *Let* $P(\partial) = \sum_{|\alpha| \leq m} C_\alpha \partial^\alpha$ *be a differential polynomial with coefficients in* $E(\Omega)$. *For any* $u \in A(\Omega)$, *we have*

$$\mathcal{S}_{\mathcal{A}}^{(a,\mathcal{F})}(P(\partial)\, u) \subset \mathcal{S}_{\mathcal{A}}^{(a,\mathcal{F})}(u).$$

Proof Since $P(\partial)$ is a continuous linear operator, the theorem is a consequence of the following lemma.

Linear and differential properties of the singular spectrum

- **Theorem 6** *Let* $P(\partial) = \sum_{|\alpha| \leq m} C_\alpha \partial^\alpha$ *be a differential polynomial with coefficients in* $E(\Omega)$*. For any* $u \in A(\Omega)$*, we have*

$$\mathcal{S}_{\mathcal{A}}^{(a,\mathcal{F})}(P(\partial)\, u) \subset \mathcal{S}_{\mathcal{A}}^{(a,\mathcal{F})}(u).$$

Proof Since $P(\partial)$ is a continuous linear operator, the theorem is a consequence of the following lemma.

- **Lemma** *Let* L *be a continuous linear operator on* $F(\Omega)$*. Then for any section* $u \in A(\Omega)$*, we have* $S_{\mathcal{A}}^{(a,\mathcal{F})}(Lu) \subset S_{\mathcal{A}}^{(a,\mathcal{F})}(u)$*.*
  Proof. First we recall that L canonically ("componentwise") extends to a linear continuous operator on $\mathcal{A}(\Omega)$. Moreover, using continuity, Lu has a limit in $\mathcal{F}$ wherever $u \in \mathcal{F}_{\mathcal{A}}$ has. Since L maps $\mathcal{F}(\Omega)$ into itself, we have $a(r)\, u \in \mathcal{F}_{\mathcal{A}}(\Omega) \;\Rightarrow\; L(a(r)u) \in \mathcal{F}_{\mathcal{A}}(\Omega)$, and by linearity, $L(a(r)u) = a(r)Lu$. Therefore, $r \in N_x(u) \;\Rightarrow\; r \in N_x(Lu)$, and thus the Theorem.

Theorem 7 *For given u and $v \in A(\Omega)$, let D_i $(i = 1, 2, 3)$ be the following disjoint sets:*

$$D_1 = \mathcal{S}_{\mathcal{A}}^{\mathcal{F}}(u) \diagdown D_3 \ ; \quad D_2 = \mathcal{S}_{\mathcal{A}}^{\mathcal{F}}(v) \diagdown D_3; \quad D_3 = \mathcal{S}_{\mathcal{A}}^{\mathcal{F}}(u) \cap \mathcal{S}_{\mathcal{A}}^{\mathcal{F}}(v).$$

Then, the (a, F)–singular asymptotic spectrum of $u\,v$ satisfies

$$\begin{aligned} \mathcal{S}_{\mathcal{A}}^{(a,\mathcal{F})}(uv) \subset & \left\{(x, r) \in D_1 \times \mathbb{R}_+^n, r \in \Sigma_x(u)\right\} \\ & \cup \left\{(x, r) \in D_2 \times \mathbb{R}_+^n, r \in \Sigma_x(v)\right\} \\ & \cup \left\{(x, r) \in D_3 \times \mathbb{R}_+^n, r \in E_x(u, v)\right\} \ , \end{aligned}$$

where for any $x \in D_3$,

$$E_x(u, v) = \mathbb{R}_+^n \setminus (N_x(u) + N_x(v))$$

Conic asymptotic regular fiber

We can add some hypotheses rendering the asymptotic singular fiber the complement of a conic subset of $\mathbb{R}_{+}^{n}$.

Theorem 8 *Consider*
$\Lambda = \left]0,1\right]^{n} \ni \lambda = (\lambda_1, \lambda_2, ..\lambda_n)$; *A overgenerated by the set* $B_n = \left\{ (\lambda_1)_\lambda , (\lambda_2)_\lambda , ..., (\lambda_n)_\lambda \right\}$; *the map a from* R_{+}^{n} *to* A_{+} *defined by* $a(r) = a(r_1, r_2, ..., r_n) = (\lambda_1^{r_1} \lambda_2^{r_2} \cdots \lambda_n^{r_n})$ *and let* $u \in A(\Omega)$ *where* Ω *is an open set in* X, *such that*

$$\exists p = (p_1, p_2, ..p_n) \in \mathbb{R}_{+}^{n}, \exists g \in \mathcal{F}(\Omega), \; g \neq 0, \; \lim_{\Lambda} {}_{\mathcal{F}(V)} a_\lambda(p) u_\lambda = g.$$

Then, for each $x \in g$, *the* $(a, \mathcal{F})$*-regular fiber is exactly the closed cone*

$$N_x(u) = \prod_{i=1}^{n} [p_i, +\infty)$$

with p as apex. It follows that the the $(a, \mathcal{F})$*–singular fiber is*

$$\Sigma_x(u) = \bigcup_{i=1,2,..n} \left\{ r \in \mathbb{R}_{+}^{n} \mid r_i \in [0, p_i) \right\}.$$

Application to a transport equation

- Consider the Cauchy problem for a linear transport equation where the coefficients α and β and the data u_0 are irregular as distributions

$$(P) \quad \begin{cases} \dfrac{\partial}{\partial t}u(t,x,y) + \alpha(t)\,\dfrac{\partial}{\partial x}u(t,x,y) + \beta(t)\,\dfrac{\partial}{\partial y}u(t,x,y) = 0\ , \\ \hfill u(0,x,y) = u_0(x,y)\ . \end{cases}$$

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- If $\alpha,\beta\in\mathrm{C}^\infty(\mathbb{R})$, $u_0\in\mathrm{C}^\infty(\mathbb{R}^2)$, $f(t)=\int_0^t\alpha(\tau)\,\mathrm{d}\tau$, $g(t)=\int_0^t\beta(\tau)\,\mathrm{d}\tau$, the smooth solution $u(t,x,y)=u_0(x-f(t),y-g(t))$ propagates the data along the caracteristic curve $\Gamma=\{t=s,x=f(s),y=g(s)\}$. If $u_0=v\in\mathcal{D}'(\mathbb{R}^2)$, and always $\alpha,\beta\in\mathrm{C}^\infty(\mathbb{R})$, The problem is well posed in $\mathcal{D}'(\mathbb{R}^3)$ and admits the solution distribution v_Γ defined for $\psi\in\mathcal{D}(\mathbb{R}^3)$ by $\langle v_\Gamma,\psi\rangle=\langle v,(x,y)\mapsto\int\psi(t,x+f(t),y+g(t))\,\mathrm{d}t\rangle$ In particular, if $v=\delta$, the solution is the Dirac measure on the characteristic curve $\langle\delta_\Gamma,\psi\rangle=\int\psi(t,f(t),g(t))\,\mathrm{d}t$

Coefficients depending on all variables

We could consider coefficients depending on all variables ie replace $\alpha(t)$ and $\beta(t)$ by $\alpha(t,x;y)$ and $\beta(t,x;y)$. But then, even in the smooth case, the solution cannot be formulated explicitely and this leads to a more difficult analysis which needs the help of characteristic method, study in progress.

Now in the case where the coefficients are distributions and the data a more singular object $\delta_x^p \otimes \delta_y^q$ to which we will give later a generalized meaning, the problem is formally written as

$$(P_{form}) \begin{cases} \dfrac{\partial}{\partial t}u + (\alpha \otimes 1_{xy})\dfrac{\partial}{\partial x}u + (\beta \otimes 1_{xy})\dfrac{\partial}{\partial y}u = 0, \\ u\mid_{\{t=0\}} = \delta_x^p \otimes \delta_y^q. \end{cases}$$

The coefficients $A = \alpha \otimes 1_{xy}$ and $B = \beta \otimes 1_{xy}$ are now distributions in $\mathcal{D}'(\mathbb{R}^3)$ but generally the factor $A\frac{\partial}{\partial x}u + B\frac{\partial}{\partial y}u$ has no meaning in $\mathcal{D}'(\mathbb{R}^3)$, and it is the same for $\delta_x^p \otimes \delta_y^q$ except for $p = q = 1$.

- We can associate to (P_{form}) a generalized problem, well formulated in a convenient $(\mathcal{C},\mathcal{E},\mathcal{P})$-algebra $\mathcal{A}(\mathbb{R}^3)$, as follows. Take $\varphi, \psi, \theta \in \mathcal{D}(\mathbb{R})$ such that $\int \varphi(x)dx = \int \psi(y)\,\mathrm{d}y = \int \theta(t)\,\mathrm{d}t = 1$. Then let $\varphi_\varepsilon = \frac{1}{\varepsilon}\varphi\left(\frac{\cdot}{\varepsilon}\right)$, $\psi_\eta = \frac{1}{\eta}\psi\frac{\cdot}{\eta}$, $\theta_\rho = \frac{1}{\rho}\psi\frac{\cdot}{\rho}$. Now set $\alpha_\rho = \alpha * \theta_\rho$, $\beta_\rho = \beta * \theta_\rho$. We consider $\mathcal{C} = A/I$ and the associated spaces $\mathcal{A}(\mathbb{R}^3)$ and $\mathcal{A}(\mathbb{R}^2)$ overgenerated by $\left\{(\varepsilon)_{\varepsilon,\eta,\rho}, (\eta)_{\varepsilon,\eta,\rho}, (\rho)_{\varepsilon,\eta,\rho}\right\}$ .Finally, we define $H_{\varepsilon,\eta}(x,y) = \varphi_\varepsilon^p(x)\,\psi_\eta^q(y)$, $F_\rho = \alpha_\rho \otimes 1_{xy}$, $G_\rho = \beta_\rho \otimes 1_{xy}$.

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- It is clear that $\left(F_\rho\right)_{\varepsilon,\eta,\rho}$ (resp. $\left(G_\rho\right)_{\varepsilon,\eta,\rho}$) is a representative of some F (resp. G) belonging to $\mathcal{A}\left(\mathbb{R}^3\right)$ and $\left(H_{\varepsilon,\eta}\right)_{\varepsilon,\eta,\rho}$ is a representative of some H belonging to $\mathcal{A}\left(\mathbb{R}^2\right)$. Thus F and G (resp. H) are the classes in $\mathcal{A}\left(\mathbb{R}^3\right)$ (resp. $\mathcal{A}\left(\mathbb{R}^2\right)$) of the families regularizing the coefficients (resp. the data). Then

$$(P_{gen})\begin{cases}\dfrac{\partial}{\partial t}u+F\,\dfrac{\partial}{\partial x}u+G\,\dfrac{\partial}{\partial y}u=0,\\ u\mid_{\{t=0\}}=H.\end{cases}$$

is well formulated.

- To solve (P_{gen}) we first solve

$$(P_\infty)\left\{\begin{array}{c} \dfrac{\partial}{\partial t}u_{\varepsilon,\eta,\rho}+a_\rho(t)\,\dfrac{\partial}{\partial x}u_{\varepsilon,\eta,\rho}+b_\rho(t)\,\dfrac{\partial}{\partial y}u_{\varepsilon,\eta,\rho}=0\ , \\ u_{\varepsilon,\eta,\rho}(0,x,y)=H_{\varepsilon,\eta}(x,y)\ , \end{array}\right.$$

with solution

$$u_{\varepsilon,\eta,\rho}(t,x,y)=H_{\varepsilon,\eta}\left(x-\Phi_\rho(t),y-\Psi_\rho(t)\right)$$

where $\Phi_\rho(t)=\int_0^t a_\rho(\tau)\,\mathrm{d}\tau$, $\Psi_\rho(t)=\int_0^t b_\rho(\tau)\,\mathrm{d}\tau$.
And we obtain

- To solve (P_{gen}) we first solve

$$(P_\infty)\begin{cases} \dfrac{\partial}{\partial t}u_{\varepsilon,\eta,\rho} + a_\rho(t)\,\dfrac{\partial}{\partial x}u_{\varepsilon,\eta,\rho} + b_\rho(t)\,\dfrac{\partial}{\partial y}u_{\varepsilon,\eta,\rho} = 0\ , \\ u_{\varepsilon,\eta,\rho}(0,x,y) = H_{\varepsilon,\eta}(x,y)\ , \end{cases}$$

  with solution

$$u_{\varepsilon,\eta,\rho}(t,x,y) = H_{\varepsilon,\eta}\left(x - \Phi_\rho(t), y - \Psi_\rho(t)\right)$$

  where $\Phi_\rho(t) = \int_0^t a_\rho(\tau)\,\mathrm{d}\tau$, $\Psi_\rho(t) = \int_0^t b_\rho(\tau)\,\mathrm{d}\tau$.
  And we obtain

- **Theorem 9** *The class of* $\left(u_{\varepsilon,\eta,\rho}\right)_{\varepsilon,\eta,\rho}$ *in* $\mathcal{A}\left(\mathbb{R}^3\right)$ *is solution of* (P_{gen})

Theorem 10 (Asymptotic singular support)
Suppose that $\lim\limits_{\rho\to0}\Phi_\rho=\Phi$, $\lim\limits_{\rho\to0}\Psi_\rho=\Psi$ *in the* $L^1_{loc}(\mathbb{R})$ *topology*
(i) *If* $p=1$ *and* $q=1$, *the* D'*–singular support of* $u=\left[u_{\varepsilon,\eta,\rho}\right]$ *is the empty set.*
(ii) *If* $p>1$ *or* $q>1$, *the* D'*-singular support of* $u=\left[u_{\varepsilon,\eta,\rho}\right]$ *is the closure* $\overline{\Gamma}$ *of the "regularized characteristic" curve*

$$\Gamma=\left\{(t,x,y):x=\Phi(t),y=\Psi\left(t\right)\right\}.$$

Theorem 11 (Asymptotic singular spectrum)
For $r = (r_1, r_2, r_3)$, *choose* $a(r) = [(\varepsilon^{r_1}\eta^{r_2}\rho^{r_3})_{(\varepsilon,\eta,\rho)}]$.
(i) *If* $p = 1$ *and* $q = 1$, *then the* (a, D')*-singular spectrum of* $u \in A(R^3)$ *is the empty set.*
(ii) *If* $p > 1$ *and* $q > 1$, *then the* (a, D')*-singular spectrum of* $u \in A(R^3)$ *is the set*

$$\mathcal{S}_{\mathcal{A}}^{(a,\mathcal{D}')}(u) = \left\{(X, r) \in \mathbb{R}^3 \times \mathbb{R}_+^3 \mid X = (t, x, y) \in \Gamma,\ r \in \Sigma_X(u)\right\} ,$$
$$\Sigma_X(u) = \{\, r \in \mathbb{R}_+^3 \mid r_1 \in [0, p-1[\ \vee \ r_2 \in [0, q-1[\,\} .$$

The singularities of the data propagate along the "regularized characteristic" Γ on which the fiber $\Sigma_X(u)$ is constant.

Bibliography

• N. Caroff, *Generalized solutions of linear partial differential equations with discontinuous coefficients.* Differential Integral Equations **17**/5-6, 653-668, (2004).

• G. M. Constantine, H. Savits, *A multivariate Faa di Bruno formula with applications*. Trans. of the A. M. S. **348**(2) (1996), 503–520.

• A. Delcroix, V. Dévoué, J.-A. Marti, *Generalized solutions of singular differential problems. Relationship with classical solutions.* J. Math. Anal. Appl. **353**, 386–402, (2009).

• A. Delcroix, J.-A. Marti , M. Oberguggenberger, *Spectral Asymptotic Analysis in Algebras of Generalized Functions*. Asymptotic Analysis 59/1-2, 83–107, (2008). DOI:10.3233/ASY-2008-0885

• V. Dévoué, M.F. Hasler, J.-A. Marti, *Multidimensioal asymptotic spectral analysis and applications.* Applicable Analysis, 90:11, 1729-1746, (2011).

• M. F. Hasler, *Asymptotic extension of topological modules and algebras.* Int. Trans. Spec. Funct., **20**/3-4, 291-299, (2009).

• C. Le Bris, P.-L. Lions, *Existence and Uniqueness of solutions to Fokker-Planck type Equations with Irregular Coefficients*. Comm. in PDE, 33/7, 1272-1317 (2008).

• J.-A. Marti, (C, E, P)-Sheaf structure and applications, in: Nonlinear Theory of Generalized Functions, Chapman & Hall/CRC Research Notes in Mathematics **401**, M. Grosser, G. Hörmann, M. Kunzinger and M. Oberguggenberger, eds., Boca Raton (1999), pp. 175–186.

• J.-A. Marti, *Nonlinear algebraic analysis of delta shock wave solutions to Burgers' equation.* Pacific Journal of Mathematics **210**, 165–187 (2003).

• J.-A. Marti, *Regularity, Local and Microlocal Analysis in Theories of Generalized Functions*. Act. Appl. Math. **105**, 267–302 (2009).

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